FUNDAMENTAL EQUATIONS OF THE THEORY OF ASYMMETRIC ELASTICITY

(OSNOVNYE URAVNENIIA TEORII NESIMMETRICHNOI UPRUGOSTI)

PMM Vol.28, № 3, 1964, pp.401-408

V.A.PAL'MOV
(Leningrad)

(Received February 13, 1964)

1. In order to explain certain laws governing the propagation of short acoustic waves in cristals, polycristalline metals and high polymers, it is essential to take account of the discrete nature of their structure, consisting as it does of sepatate particles held together by complex forces of interaction. These particles may be individual molecules, individual crystals in polycrystalline metals, etc. The essential difference between the continuous medium usually considered in the theory of elasticity and the real system of separate particles lies in the following. The displacement of particles in a continuous medium can be specified by a vector field $\mathbf{u}$, and a small rotation, which when $\mathbf{u}$ is small may be found from the formula

$$\mathbf{\Phi} = {}^1/_2 \operatorname{rot} \mathbf{u} \tag{1.1}$$

If we treat the material as a system of discrete particles the displacement of their centers of gravity can be defined by a vector field $\mathbf{u}$, and a small rotation about the center of gravity by a vector field $\mathbf{\Phi}$, which is kinematically independent of $\mathbf{u}$.

Furthermore, in the theory of a continuous medium the action of the medium on a small element isolated from it is determined by the stresses, or what amounts to the same, by the forces acting on its faces, and the moment is calculated in terms of these forces. However, if we treat a medium as a system of discrete particles, the action on one particle from its neighboring particles is determined by independent forces and moments.

In what follows we shall approach the study of the behavior of a medium with a discrete structure on the basis of the theory of a continuous medium. In order to avoid the above differences between the classical continuous medium and the system of discrete particles we assume the continuous medium to posses a number of properties which at first may appear somewhat unusual.

We define small displacements of particles in the continuous medium by a vector field $\mathbf{u}$ and a small rotation of these particles by a vector field $\mathbf{\Phi}$, which is independent of $\mathbf{u}$. The state of stress at any point in the continuous medium will be defined by the stress diadic τ and the diadic of couple-stresses μ. The elements of the stress diadic are the forces acting on the unit areas of the appropriate sections within the body. The elements of the diadic of couple-stresses are the moments acting on unit areas of the same sections. Body forces at any point in the medium will be specified by the force vector $\mathbf{K}$ and the body-moment vector $\mathbf{c}$.

It is of importance in the subsequent theory to make the assumption that the surface and body forces do work only in the virtual displacements $\delta\mathbf{u}$, and that the surface and body moments do work only in the virtual displacements $\delta\mathbf{\Phi}$. A similar approach to these problems may be found in [1 to 7]. However, in a number of works [2 to 5] the kinematic hypothesis (1.1) is retained side by side with the introduction of couple-stresses.

2. Let us isolate from the medium a volume V having a surface area S. In accordance with the above, at every point on the surface S the action of the part of the medium situated outside S on the part inside S is given by the stress vector τ_n and the vector of couple-stresses μ_n; at every point in the volume the body forces and moments have intensities $\mathbf{K}$ and $\mathbf{c}$ respectively.

For the isolated volume of the medium to be in equilibrium it is necessary and sufficient for the following conditions to be satisfied:

$$\int_S \boldsymbol{\tau}_n dS + \int_V \mathbf{K}\, dV = 0 \qquad (2.1)$$

$$\int_S (\mathbf{r}\times\boldsymbol{\tau}_n + \boldsymbol{\mu}_n)\, dS + \int_V (\mathbf{r}\times\mathbf{K} + \mathbf{c})\, dV = 0 \qquad (2.2)$$

We note that the following relations hold for the vectors of surface loading τ_n and μ_n [4 and 8]:

$$\boldsymbol{\tau}_n = \mathbf{n}\cdot\boldsymbol{\tau}, \qquad \boldsymbol{\mu}_n = \mathbf{n}\cdot\boldsymbol{\mu} \qquad (2.3)$$

where $\mathbf{n}$ is the unit normal to the surface S, whereas τ and μ are the diadics of stresses and couple-stresses.

Substituting (2.3) into (2.1) and (2.2) and transforming the surface integrals by the Gauss-Ostrogradskii formula, we obtain

$$\int_S \boldsymbol{\tau}_n dS + \int_V \mathbf{K}\, dV = \int_S \mathbf{n}\cdot\boldsymbol{\tau}\, dS + \int_V \mathbf{K}\, dV = \int_V (\nabla\cdot\boldsymbol{\tau} + \mathbf{K})\, dV = 0 \qquad (2.4)$$

$$\int_S (\mathbf{r}\times\boldsymbol{\tau}_n + \boldsymbol{\mu}_n)\, dS + \int_V (\mathbf{r}\times\mathbf{K} + \mathbf{c})\, dV = \int_S \mathbf{n}\cdot(-\boldsymbol{\tau}\times\mathbf{r} + \boldsymbol{\mu})\, dS +$$
$$+\int_V (\mathbf{r}\times\mathbf{K} + \mathbf{c})\, dV = \int_V [\nabla\cdot(-\boldsymbol{\tau}\times\mathbf{r} + \boldsymbol{\mu}) + \mathbf{r}\times\mathbf{K} + \mathbf{c}]\, dV = 0 \quad (2.5)$$

where ∇ represents the Hamilton differential operator $\nabla = \partial / \partial\mathbf{r}$. Hence, since the volume V is arbitrary, we obtain the differential equations of equilibrium

$$\nabla\cdot\boldsymbol{\tau} + \mathbf{K} = 0 \quad (2.6)$$

$$\nabla\cdot(-\boldsymbol{\tau}\times\mathbf{r} + \boldsymbol{\mu}) + \mathbf{r}\times\mathbf{K} + \mathbf{c} = 0 \quad (2.7)$$

In order to simplify (2.7) we use the formula [8]

$$\nabla\cdot(\boldsymbol{\tau}\times\mathbf{r}) = (\nabla\cdot\boldsymbol{\tau})\times\mathbf{r} - \boldsymbol{\tau}_\times \quad (2.8)$$

where $\boldsymbol{\tau}_\times$ is the vector of the diadic $\boldsymbol{\tau}$. We recall that we can obtain this by vectorially multiplying the left-hand factors of the diadic by the right-hand, and adding the results.

If we substitute (2.8) into (2.7) we obtain

$$\mathbf{r}\times(\nabla\cdot\boldsymbol{\tau} + \mathbf{K}) + \nabla\cdot\boldsymbol{\mu} + \boldsymbol{\tau}_\times + \mathbf{c} = 0 \quad (2.9)$$

which, on the basis of (2.6) can be considerably simplified to become

$$\nabla\cdot\boldsymbol{\mu} + \boldsymbol{\tau}_\times + \mathbf{c} = 0 \quad (2.10)$$

Equations (2.6) and (2.10) are the required equations of equilibrium. Note that the stress diadic $\boldsymbol{\tau}$ is asymmetrical since, from (2.10), its vector is not zero. This is the explanation of the title – the theory of asymmetric elasticity [2].

3. In order to find the relation between the kinematic quantities $\mathbf{u}$ and $\boldsymbol{\Phi}$ and the force quantities $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ we use the principle of of virtual displacements. Assuming the existence of potential energy of elastic deformation of the medium, the density of which, U, depends on the field of small displacements $\mathbf{u}$ and the field of rotations $\boldsymbol{\Phi}$, we have

$$\int_S (\boldsymbol{\tau}_n\cdot\delta\mathbf{u} + \boldsymbol{\mu}_n\cdot\delta\boldsymbol{\Phi})\, dS + \int_V (\mathbf{K}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\delta\boldsymbol{\Phi} - \delta U)\, dV = 0 \quad (3.1)$$

Here $\delta\mathbf{u}$ is a field of virtual displacements of particles within the medium and $\delta\boldsymbol{\Phi}$ is a field of virtual particle rotations. Substituting (2.3) into (3.1) and transforming the surface integral into a volume integral, we obtain

$$\int_V [\nabla\cdot(\boldsymbol{\tau}\cdot\delta\mathbf{u} + \boldsymbol{\mu}\cdot\delta\boldsymbol{\Phi}) + \mathbf{K}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\delta\boldsymbol{\Phi} - \delta U]\, dV = 0 \quad (3.2)$$

Since the volume V is arbitrary we can set the expression under the integral in (3.2) equal to zero, from which we can find the variation in the density of potential energy

$$\delta U = \nabla\cdot(\boldsymbol{\tau}\cdot\delta\mathbf{u} + \boldsymbol{\mu}\cdot\delta\boldsymbol{\Phi}) + \mathbf{K}\cdot\delta\mathbf{u} + \mathbf{c}\cdot\delta\boldsymbol{\Phi} \tag{3.3}$$

Further, we can make use of the formulas

$$\nabla\cdot(\boldsymbol{\tau}\cdot\delta\mathbf{u}) = (\nabla\cdot\boldsymbol{\tau})\cdot\delta\mathbf{u} + \boldsymbol{\tau}\cdot\cdot\,\delta\nabla\mathbf{u}^* \tag{3.4}$$

$$\nabla\cdot(\boldsymbol{\mu}\cdot\delta\boldsymbol{\Phi}) = (\nabla\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{\Phi} + \boldsymbol{\mu}\cdot\cdot\,\delta\nabla\boldsymbol{\Phi}^* \tag{3.5}$$

The second term in the right-hand side of (3.4) is the double scalar product of the diatics $\boldsymbol{\tau}$ and $\delta\nabla\mathbf{u}^*$. In such a multiplication the right- and left-hand factors respectively of the diadics of the cofactors multiply scalarly. Substituting (3.4) into (3.3) we obtain

$$\delta U = (\nabla\cdot\boldsymbol{\tau} + \mathbf{K})\cdot\delta\mathbf{u} + (\nabla\cdot\boldsymbol{\mu} + \mathbf{c})\cdot\delta\boldsymbol{\Phi} + \boldsymbol{\tau}\cdot\cdot\,\delta\nabla\mathbf{u}^* + \boldsymbol{\mu}\cdot\cdot\,\delta\nabla\boldsymbol{\Phi}^* \tag{3.6}$$

Making use of the equilibrium equations we can reduce this expression to the form

$$\delta U = -\,\boldsymbol{\tau}_{\times}\cdot\delta\boldsymbol{\Phi} + \boldsymbol{\tau}\cdot\cdot\,\delta\nabla\mathbf{u}^* + \boldsymbol{\mu}\cdot\cdot\,\delta\nabla\boldsymbol{\Phi}^* \tag{3.7}$$

Corresponding to the vector $\boldsymbol{\Phi}$, we introduce an antisymmetric diadic $\boldsymbol{\Phi}^A$, whose vector is equal to $\boldsymbol{\Phi}$. A direct check shows that $\boldsymbol{\Phi}^A$ is defined by the formula

$$\boldsymbol{\Phi}^A = -\,{}^1/_2\,\mathbf{I}\times\boldsymbol{\Phi} \tag{3.8}$$

where $\mathbf{I}$ is the unit diadic.

By definition we have

$$\boldsymbol{\Phi}_{\times}{}^{A} = \boldsymbol{\Phi} \tag{3.9}$$

Similarly, we can establish the antisymmetric component of the stress diadic from the value of $\boldsymbol{\tau}_{\times}$:

$$\boldsymbol{\tau}^A = -\,{}^1/_2\,\mathbf{I}\times\boldsymbol{\tau}_{\times} \tag{3.10}$$

Further, a direct check shows that the following relation holds

$$\boldsymbol{\tau}_{\times}\cdot\delta\boldsymbol{\Phi} = -\,2\boldsymbol{\tau}^A\cdot\cdot\,\delta\boldsymbol{\Phi}^A = -\,2\boldsymbol{\tau}\cdot\cdot\,\delta\boldsymbol{\Phi}^A \tag{3.11}$$

Substituting (3.11) into (3.7) and introducing the notation

$$\boldsymbol{\Lambda} = \nabla\mathbf{u} - 2\boldsymbol{\Phi}^A, \qquad \mathbf{M} = \nabla\boldsymbol{\Phi} \tag{3.12}$$

we can write the variation of potential energy in the form

$$\delta U = \boldsymbol{\tau}\cdot\cdot\,\delta\boldsymbol{\Lambda}^* + \boldsymbol{\mu}\cdot\cdot\,\delta\mathbf{M}^* \tag{3.13}$$

It follows that the specific potential energy of small deformations may be expressed as a function of the diadics $\boldsymbol{\Lambda}$ and $\mathbf{M}$

$$U = U\,(\boldsymbol{\Lambda}, \mathbf{M}) \tag{3.14}$$

The variation of this is

$$\delta U = \frac{\partial U}{\partial\boldsymbol{\Lambda}}\cdot\cdot\,\delta\boldsymbol{\Lambda}^* + \frac{\partial U}{\partial\mathbf{M}}\cdot\cdot\,\delta\mathbf{M}^* \tag{3.15}$$

Comparing (3.15) and (3.13), we arrive at the general expression for the law of elastic deformation

$$\boldsymbol{\tau} = \frac{\partial U}{\partial \boldsymbol{\Lambda}}, \qquad \boldsymbol{\mu} = \frac{\partial U}{\partial \mathbf{M}} \tag{3.16}$$

From now on we shall confine our attention to small deformations in an isotropic medium with mirror symmetry of its properties. Since the medium is isotropic the potential energy must depend only on the invariants of the diadics $\boldsymbol{\Lambda}$ and $\mathbf{M}$. But since the deformations are small it is sufficient to retain in the potential energy only second-order terms, and assume that first-order terms are eliminated by an appropriate choice of the reference origin of the displacements $\mathbf{u}$ and rotations $\boldsymbol{\Phi}$. Therefore, in deriving an expression for the density of potential energy, of the six invariants of both diadics $\boldsymbol{\Lambda}$ and $\mathbf{M}$, we can use only the first and the second scalar invariants and vector invariants. Note that the products of the first scalar or vector invariants of different diadics cannot appear in the expression for U, since the diadic $\boldsymbol{\Lambda}$ is polar and the diadic $\mathbf{M}$ is axial.

Thus the specific potential energy of the medium may be written in the form

$$U = \frac{1}{2}\frac{\lambda}{2}\boldsymbol{\Lambda}^S \cdot\cdot \mathbf{II} \cdot\cdot \boldsymbol{\Lambda}^S + \mu\boldsymbol{\Lambda}^S \cdot\cdot \boldsymbol{\Lambda}^S - \alpha\boldsymbol{\Lambda}^A \cdot\cdot \boldsymbol{\Lambda}^A + \\ + \frac{1}{2}\frac{\beta}{2}\mathbf{M}^S \cdot\cdot \mathbf{II} \cdot\cdot \mathbf{M}^S + \gamma\mathbf{M}^S \cdot\cdot \mathbf{M}^S - \varepsilon\mathbf{M}^A \cdot\cdot \mathbf{M}^A \tag{3.17}$$

where the index S indicates symmetric components of diadics and the index A indicates antisymmetric components. The coefficients λ, μ, α, β, γ and ε are the six elastic characteristics of the isotropic medium. We see that in Expression (3.17) they are multiplied by linearly independent invariants of the diadics $\boldsymbol{\Lambda}$ and $\mathbf{M}$.

Substituting (3.17) into (3.16), we obtain the law of small elastic deformations equivalent to the usual Hooke's law

$$\boldsymbol{\tau} = \lambda\mathbf{II} \cdot\cdot \boldsymbol{\Lambda}^S + 2\mu\boldsymbol{\Lambda}^S + 2\alpha\boldsymbol{\Lambda}^A \tag{3.18}$$

$$\boldsymbol{\mu} = \beta\mathbf{II} \cdot\cdot \mathbf{M}^S + 2\gamma\mathbf{M}^S + 2\varepsilon\mathbf{M}^A \tag{3.19}$$

4. Equations (2.6), (2.10), (3.12), (3.18) and (3.19) constitute the complete system of equations of the linear theory of elasticity, which takes into account rotational interaction between particles. For certain values of the parameter α this system can be reduced to the classical equations of the theory of elasticity and the equations of the theory of couple-stresses considered in [2 to 5].

For instance, it follows from (3.18) that if $\alpha = 0$ the stress diadic becomes symmetrical, and by virtue of (3.12) assumes the form

$$\boldsymbol{\tau} = \lambda\mathbf{I}\nabla\cdot\mathbf{u} + 2\mu\,(\nabla\mathbf{u})^S \tag{4.1}$$

Equations (4.1) and (2.6) form a system of equations of the classical theory of elasticity [9], in which λ and μ are the usual Lamé parameters. In this case Equations (3.19) and (2.10) comprise an independent system for the determination of the particle rotations.

Consider now the case when $\alpha \to \infty$. Since the stresses must be finite, we have

$$\lim_{\alpha\to\infty} 2\alpha\Lambda^A = \tau^A, \qquad \lim_{\alpha\to\infty} \Lambda^A = 0 \tag{4.2}$$

where τ^A is an antisymmetric diadic.

It follows from the second relation of (4.2) that

$$\lim_{\alpha\to\infty} \Lambda_\times{}^A = 0$$

which, on the basis of (3.12), gives

$$\Phi = {}^1/_2 \nabla \times \mathbf{u} \tag{4.3}$$

Substituting (3.12) and (4.3) into (3.18) and (3.19), we obtain

$$\tau = \lambda \mathbf{I} \nabla\cdot\mathbf{u} + 2\mu (\nabla\mathbf{u})^S + \tau^A$$

$$\mu = \gamma (\nabla\nabla\times\mathbf{u})^S + \varepsilon (\nabla\nabla\times\mathbf{u})^A = \frac{\gamma+\varepsilon}{2} \nabla\nabla\times\mathbf{u} + \frac{\gamma-\varepsilon}{2} \nabla\times\mathbf{u}\nabla \tag{4.4}$$

Equations (2.6), (2.10) and (4.4) have been studied in [3 to 5].

The system of equations (2.6), (2.10), (3.12), (3.18) and (3.19) must be provided with boundary conditions. If these are force conditions then surface loads $\mathbf{n}\cdot\tau$ and $\mathbf{n}\cdot\mu$ must be specified on the surface of the elastic body, when $\mathbf{n}$ is the unit normal to the surface. If the conditions are kinematic, displacements $\mathbf{u}$ and rotations Φ must be specified on the surface. If boundary conditions and body forces $\mathbf{K}$ and $\mathbf{c}$ are specified and the density of potential energy U is a positive definite quadratic form, then the solution of the above system of equations is unique. This theorem may be proved for the present case in the same way as in the calssical theory of elasticity.

Substituting (3.18), (3.19) and (3.12) into the equilibrium equations (2.6) and (2.10), we obtain equations in the displacements $\mathbf{u}$ and rotations Φ analogous to the Lamé equations

$$(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + 2\alpha\nabla\times\Phi + \mathbf{K} = 0$$

$$(\beta + 2\gamma)\nabla\nabla\cdot\Phi - (\gamma+\varepsilon)\nabla\times(\nabla\times\Phi) + 2\alpha\nabla\times\mathbf{u} - 4\alpha\Phi + \mathbf{c} = 0 \tag{4.5}$$

In the derivation of these equations the following expressions for the symmetric and antisymmetric components of the diadics Λ and $\mathbf{M}$ and for the vector invariant of the diadic τ were used

$$
\begin{aligned}
\Lambda^S &= {}^1/_2(\nabla \mathbf{u} - \mathbf{u}\nabla) + \mathbf{u}\nabla = -{}^1/_2 \mathbf{I} \times (\nabla \times \mathbf{u}) + \mathbf{u}\nabla \\
\mathbf{M}^S &= -{}^1/_2 \mathbf{I} \times (\nabla \times \mathbf{\Phi}) + \mathbf{\Phi}\nabla \\
\Lambda^A &= -{}^1/_2 \mathbf{I} \times \Lambda_\times = -{}^1/_2 \mathbf{I} \times (\nabla \times \mathbf{u} - 2\mathbf{\Phi}) \\
\mathbf{M}^A &= -{}^1/_2 \mathbf{I} \times (\nabla \times \mathbf{\Phi}) \\
\tau_\times &= 2\alpha \Lambda_\times = 2\alpha(\nabla \times \mathbf{u} - 2\mathbf{\Phi})
\end{aligned}
\tag{4.6}
$$

In addition, the elastic characteristics $\lambda, \mu, \ldots, \epsilon$ were assumed to be constants.

5. Let us consider the propagation of waves in an infinite dynamically isotropic elastic medium. In this case, for the body forces in Equations (4.5) we take the inertia forces

$$
\mathbf{K} = -\rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \qquad \mathbf{c} = -j \frac{\partial^2 \mathbf{\Phi}}{\partial t^2} \tag{5.1}
$$

Here ρ is the density of the medium and j is a special dynamic property of the medium equal to the product of the moment of inertia of a particle about an axis passing through its center of gravity and the number of particles per unit volume. Substituting (5.1) into (4.5), we find the dynamic equation of the medium

$$
\begin{aligned}
(\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} - (\mu + \alpha)\nabla \times (\nabla \times \mathbf{u}) + 2\alpha\nabla \times \mathbf{\Phi} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0 \\
(\beta + 2\gamma)\nabla\nabla\cdot\mathbf{\Phi} - (\gamma + \varepsilon)\nabla \times (\nabla \times \mathbf{\Phi}) + 2\alpha\nabla \times \mathbf{u} - 4\alpha\mathbf{\Phi} - j \frac{\partial^2 \mathbf{\Phi}}{\partial t^2} = 0
\end{aligned}
\tag{5.2}
$$

We express the solution to this system in the form

$$
\begin{aligned}
\mathbf{u} &= \nabla\varphi + \nabla \times \mathbf{H}, \qquad \nabla\cdot\mathbf{H} = 0 \\
\mathbf{\Phi} &= \nabla\psi + \mathbf{\Phi}_1, \qquad \nabla\cdot\mathbf{\Phi}_1 = 0
\end{aligned}
\tag{5.3}
$$

Substitution of (5.3) into (5.2) yields

$$
\begin{aligned}
&\nabla\left[(\lambda + 2\mu)\nabla^2\varphi - \rho \frac{\partial^2 \varphi}{\partial t^2}\right] + \nabla \times \left[(\mu + \alpha)\nabla^2\mathbf{H} + 2\alpha\mathbf{\Phi}_1 - \rho \frac{\partial^2 \mathbf{H}}{\partial t^2}\right] = 0 \\
&\nabla\left[(\beta + 2\gamma)\nabla^2\psi - 4\alpha\psi - j \frac{\partial^2 \psi}{\partial t^2}\right] + \left[(\gamma + \varepsilon)\nabla^2\mathbf{\Phi}_1 - 2\alpha\nabla^2\mathbf{H} - \right. \\
&\qquad \left. - 4\alpha\mathbf{\Phi}_1 - j \frac{\partial^2 \mathbf{\Phi}_1}{\partial t^2}\right] = 0
\end{aligned}
\tag{5.4}
$$

We see that the dynamic equations of the medium will be satisfied completely if $\varphi, \psi, \mathbf{H}$ and $\mathbf{\Phi}_1$ satisfy Equations

$$
(\lambda + 2\mu)\nabla^2\varphi - \rho \frac{\partial^2 \varphi}{\partial t^2} = 0, \qquad (\beta + 2\gamma)\nabla^2\psi - 4\alpha\psi - j \frac{\partial^2 \psi}{\partial t^2} = 0 \tag{5.5}
$$

$$
(\mu + \alpha)\nabla^2\mathbf{H} + 2\alpha\mathbf{\Phi}_1 - \rho \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{5.6}
$$

$$
(\gamma + \varepsilon)\nabla^2\mathbf{\Phi}_1 - 2\alpha\nabla^2\mathbf{H} - 4\alpha\mathbf{\Phi}_1 - j \frac{\partial^2 \mathbf{\Phi}_1}{\partial t^2} = 0 \tag{5.7}
$$

The first of Equations (5.5) determines the behavior of an expansion wave in the medium and the second, a rotation wave in which the particles undergo rotation but not translation. Equations (5.6) and (5.7) define distortion waves. If we eliminate the vector $\mathbf{\Phi}_1$ from these equations, we obtain a single equation for the distortion wave

$$\left\{\left[(\gamma+\varepsilon)\nabla^2-4\alpha-j\frac{\partial^2}{\partial t^2}\right]\left[(\mu+\alpha)\nabla^2-\rho\frac{\partial^2}{\partial t^2}\right]+4\alpha^2\nabla^2\right\}\mathbf{H}=0 \quad (5.8)$$

Let us consider each of the waves separately. We shall confine our attention to plane waves, propagating, say, along the x-axis. We assume

$$\varphi = Ae^{i\xi(x-ct)} = Ae^{i(\xi x-\omega t)} \quad (5.9)$$

Here A is the wave amplitude, ξ the wave number, c the phase velocity and ω the frequency of oscillations of particles in the wave. We substitute (5.9) into the first of Equations (5.5) to find the phase velocity of the wave

$$c^2 = c_1{}^2 = \frac{\lambda+2\mu}{\rho} \quad (5.10)$$

The same result is obtained in the classical theory of elasticity.

We now substitute a solution of the type (5.9) into the second of Equations (5.5) to find the relation between the phase velocity and the wave frequency ω. We obtain

$$c^2=\frac{\omega^2c_5{}^2}{\omega^2-\omega_*{}^2} \qquad \left(c_5{}^2=\frac{\beta+2\gamma}{j},\ \omega_*{}^2=\frac{4\alpha}{j}\right) \quad (5.11)$$

It follows from (5.11) that a travelling rotation wave can exist only at frequencies higher than ω_*. As $\omega\to\infty$ the phase velocity tends to c_5.

In order to investigate the behavior of distortion waves, we set

$$\mathbf{H}=\mathbf{B}e^{i\xi(x-ct)}=\mathbf{B}e^{i(\xi x-\omega t)} \quad (5.12)$$

where $\mathbf{B}$ is a constant vector which defines the direction and intensity of motion of the particles. Substituting (5.12) into (5.8) we obtain the following equation for the square of the wave number:

$$c_3{}^2c_4{}^2(\xi^2)^2+[\omega_*{}^2c_2{}^2-\omega^2(c_3{}^2+c_4{}^2)](\xi^2)-\omega^2(\omega_*{}^2-\omega^2)=0 \quad (5.13)$$

where

$$c_2{}^2=\frac{\mu}{\rho},\qquad c_3{}^2=\frac{\mu+\alpha}{\rho},\qquad c_4{}^2=\frac{\gamma+\varepsilon}{j},\qquad \omega_*{}^2=\frac{4\alpha}{j} \quad (5.14)$$

The discriminant of equation (5.13) may be written as

$$D=[\omega^2(c_3{}^2-c_4{}^2)-\omega_*{}^2c_2{}^2]^2+4\omega^2\omega_*{}^2c_4{}^2(c_3{}^2-c_2{}^2) \quad (5.15)$$

It follows that it is positive, and therefore for any ω Equation (5.13) has two different real roots ξ^2. By means of Vieta's theorem it can easily be proved that for $\omega<\omega_*$ Equation (5.13) has one positive root ξ^2, and for $\omega > \omega_*$ both roots are positive. Consequently, for $\omega < \omega_*$ there exists one travelling distortion wave and for $\omega > \omega_*$ there exist two such waves.

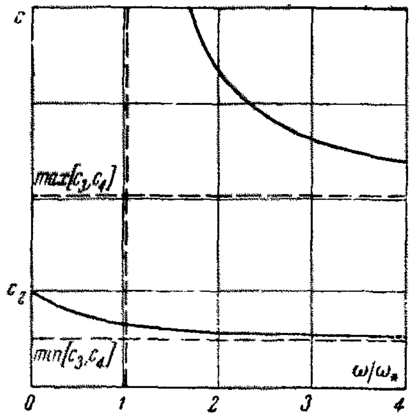

Fig. 1

For small frequencies we have the following approximate value for the positive root of Equation (5.13):

$$\xi^2 \approx \frac{\omega^2}{c_2^2} \qquad (5.16)$$

We see from,this that for low frequencies the phase velocity of the distortion wave is approximately equal to c_2, which coincides with the classical result. For high frequencies an asymptotic solution of Equation (5.13) gives two values for the wave number

$$\xi_1^2 = \frac{\omega^2}{c_3^2}, \qquad \xi_2^2 = \frac{\omega^2}{c_4^2} \qquad (5.17)$$

Hence we obtain two phase velocities for distortion waves c_3 and c_4.

From the foregoing we may conclude that the relation between the phase velocity of a distortion wave and frequency must be of the form expressed in Fig.1. This figure shows that for $\omega < \omega_*$ the phase velocity increases with increase in ω if $c_2 < \min[c_3, c_4]$ and decreases if $c_2 > \min[c_3, c_4]$.

This conclusion is at variance with the assertion of Mindlin [4] that the phase velocity of a distortion wave must increase under any conditions with increase in wave frequency.

BIBLIOGRAPHY

1. Cosserat, E and F., Théorie des Corps Deformables, Paris, 1909.
2. Grioli, G., Elasticita asimmetrica. Ann.mat.pura ed appl., Ser.IV,Vol.50, 1960.
3. Aero, E.L. and Kuvshinskii, E.V., Osnovnye urvneniia teorii uprugosti sred s vrashchatel'nym vzaimodeistviem chastits (Fundamental equations of the theory of elasticity for media with rotational interaction of particles). Fizika tverdogo tela (Physics of solids), Vol.2, № 7,1960.
4. Mindin, R.D. and Tiersten, H.F., Effects of couple-stresses in linear elasticity, Arch.Rat.Mech.Anal., Vol.11, № 5, 1962.
5. Toupin, R.A., Elastic materials with couple-stresses. Arch.Rat.Mech. Anal., Vol.11, № 5, 1962.
6. Schaefer, H., Versuch einer Elastizitätstheorie des zweidimensionalen ebenen Cosserat-Kontinuums. Miszellanen der Angewandten Mechanik. Festschrift W. Tollmien, 1962.
7. Kuvshinskii E.V. and Aero, E.L., Kontinual'naia teoriia asimmetricheskoi uprugosti. Uchet vnutrennego vrashcheniia (Continuum theory of asymmetric elasticity with account of internal rotation). Fizika tverdogo tela (Physics of solids), Vol.5, № 9, 1963.
8. Lagalli, M., Vektornoe ischislenie (Vector Analysis). Gostechizdat,1936.

9. Lur'e, A.I., Prostranstvennye zadachi teorii uprugosti (Three-dimensional Problems of the Theory of Elasticity). Gostekhizdat, 1955.

Translated by J.K.L.